Characteristic polynomials of random matrices at edge singularities

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We have discussed earlier the correlation functions of the random variables $\det(\lambda - X)$ in which X is a random matrix. In particular, the moments of the distribution of these random variables are universal functions, when measured in the appropriate units of the level spacing. When the λ's, instead of belonging to the bulk of the spectrum, approach the edge, a crossover takes place to an Airy or to a Bessel problem, and we consider here these modified classes of universality. Furthermore, when an external matrix source is added to the probability distribution of X, various different phenomenona may occur and one can tune the spectrum of this source matrix to other critical points. Again there are remarkably simple formulas for arbitrary source matrices, which allow us to compute the moments of the characteristic polynomials in these cases as well.

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I. INTRODUCTION

In the theory of random matrices, the n-point correlation functions of the eigenvalues are known to be expressible as the determinant of a two-point kernel [1,2]. The expression for this kernel depends on the various classes of universality: it is a simple sine kernel within the bulk of unitary invariant ensembles, an Airy kernel at the edge of the spectrum, or a Bessel kernel for other invariance properties of the measure. The level spacing probability $p(s)$ has also been computed recently for those different kernels [2,3].

Another interesting object is given by the average moments of the characteristic polynomial of the random matrix. These characteristic polynomials were first investigated in [4,5] for a uniform probability measure on unitary matrices, in connection with the moments of the Riemann zeta function. These results have been generalized to random Hermitian $N\times N$ matrices X with a unitary invariant probability measure

$$P(X)=\frac{1}{Z}\exp - N\,\mathrm{Tr}\,V(X). \tag{1}$$

Explicit formulas for the $2K$-point functions

$$F_{2K}(\lambda_1,\ldots,\lambda_{2K})=\left\langle \prod_1^{2K}\det(\lambda_l - X)\right\rangle \tag{2}$$

have been derived, which show that these functions are universal in the Dyson limit, in which the size N of the matrices goes to infinity, the distances between the λ's go to zero, and the products $N(\lambda_i-\lambda_j)$ remain finite. In particular, the moments

$$F_{2K}(\lambda,\ldots,\lambda)=\langle[\det(\lambda - M)]^{2K}\rangle \tag{3}$$

of the distribution of the characteristic polynomials are given in the large N limit by [6,7]

$$\exp[-NKV(\lambda)]F_{2K}(\lambda,\ldots,\lambda)=[2\pi N\rho(\lambda)]^{K^2}e^{-NK}\gamma_K, \tag{4}$$

with

$$\gamma_K=\prod_0^{K-1}\frac{l!}{(K+l)!}, \tag{5}$$

provided λ belongs to the bulk of the support of the distribution of the eigenvalues, i.e., provided $\rho(\lambda)$ does not vanish. Then one sees explicitly that the only dependence upon the probability measure is through the average density of eigenvalues $\rho(\lambda)$, and even the coefficient γ_K is a universal number.

However, the result does take different forms for different universality classes. Our previous investigations for the three classical Lie groups U(N), Sp(N), and O(N) are extended here to the Bessel kernel and Airy kernel, for which the density of states $\rho(\lambda)$ presents a singularity at the edge of the spectrum. Furthermore, we have considered a Gaussian case in which an external matrix source is present [8] in the probability distribution of the matrix,

$$P(X)=\frac{1}{Z}\exp(-N\,\mathrm{Tr}\,\tfrac{1}{2}X^2+N\,\mathrm{Tr}\,AX). \tag{6}$$

Explicit and simple formulas will be derived here again for the correlation functions and the moments of the characteristic polynomials of the matrix X, which depend on the spectrum of the matrix A. By tuning the spectrum of A appropriately, one can generate a number of different situations. For instance, we have investigated in the past the case in which the average spectrum of X presents a gap in the presence of A, and by tuning A one can study the critical point at which this gap vanishes. This creates again a new class of universality, and a new kernel [9,10]. Other cases, such as two-

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dimensional (2D) gravity in the double scaling limit, or the Penner model, would certainly be of interest as well.

II. SINE KERNEL

For completeness, and for later use, we begin with the bulk unitary case, governed by the sine kernel, but with a derivation that differs from our previous one [6]. An interesting geometric interpretation of this problem will also be provided. The kernel, from which all the correlation functions may be obtained, is given in terms of orthogonal polynomials for finite N, but reduces in the Dyson large N limit to the sine kernel

$$K(x,y)=\frac{\sin(x-y)}{x-y} \tag{7}$$

in which x and y are the eigenvalues measured in the scale of the average spacing $[2\pi\rho(\lambda)N]^{-1}$. Thus one obtains the normalized moments

$$I_K=e^{-NKV(\lambda)}\frac{\langle[\det(\lambda-X)]^{2K}\rangle}{[2\pi N\rho(\lambda)]^{K^2}}=\lim_{\lambda_i\to 0}\frac{\det K(\lambda_i,\lambda_j)}{\Delta^2(\Lambda)}, \tag{8}$$

where $\Delta(\Lambda)=\Pi_{i<j}(\lambda_i-\lambda_j)$ and $i,j=1,\ldots,K$. The right-hand side may be expressed as a contour integral, following Eq. (52) of [6],

$$I_K=\oint\oint\prod_i^K\frac{du_i\,dv_i}{(2\pi i)^2}\frac{\Delta(U)\Delta(V)}{\prod_{i=1}^K u_i^K\prod_{i=1}^K v_i^K}\prod_{i=1}^K\frac{\sin(u_i-v_i)}{u_i-v_i}. \tag{9}$$

This may be further reduced to

$$I_K=\det(a_{nm}), \tag{10}$$

$$a_{nm}=\frac{1}{n!m!}\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m}\frac{\sin(u-v)}{u-v}\bigg|_{u=v=0}, \tag{11}$$

where $n,m=0,1,\ldots,K-1$. The explicit evaluation of the determinant of $a_{n,m}$ gives

$$\det(a_{nm})=2^{K^2-K}\prod_{l=0}^{K-1}\frac{l!}{(K+l)!}. \tag{12}$$

We do recover in this way the factor γ_K [Eq. (5)] (up to a factor 2^{K^2-K} due to a different normalization of the kernel).

It is quite remarkable that this universal normalizing factor γ_K has a geometric interpretation as a Fredholm determinant of the Dirac Laplacian on the two-dimensional sphere S^2. The determinant of the Laplacian has been discussed in connection with string theory [11,12], and the relation of γ_K to this Fredholm determinant of the Laplacian was noticed in [5]. Indeed, let us show that

$$\gamma_K=\frac{e^{K^2(1+\gamma)}}{\Delta^+(-K)}, \tag{13}$$

where γ is Euler's constant and $\Delta^+(z)$ the determinant of a Dirac operator, defined below. The derivation goes as follows. Let us introduce a function $G(z)$ that satisfies the functional relation

$$G(z+1)=\Gamma(z)G(z). \tag{14}$$

It is then straightforward to verify that

$$\gamma_K=\prod_{l=0}^{K-1}\frac{l!}{(K+l)!}=2^{K-2K^2}\frac{\pi^{K+1/2}}{\Gamma(K+\frac{1}{2})}\left[\frac{G(\frac{1}{2})}{G(K+\frac{1}{2})}\right]^2. \tag{15}$$

A function G satisfying the functional relation (14) is known in the literature as a Barnes function (or as the inverse of a digamma function). It is defined by

$$\begin{aligned}G(z+1)&=\frac{1}{\Gamma_2(z+1)}\\&=(2\pi)^{z/2}e^{-[z+(1+\gamma)z^2]/2}\\&\quad\times\prod_1^\infty\left[\left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)}\right].\end{aligned} \tag{16}$$

It has been noticed earlier [13] that this Barnes function is related to the Fredholm determinant of the Laplacian on S^2. Indeed, this Fredholm determinant is the (regularized) product

$$\Delta(z)=\prod_l\left(1-\frac{z}{\lambda_l}\right)^{g_l}, \tag{17}$$

where the λ_l are the eigenvalues of the Laplacian and g_l their degeneracy, i.e., $\lambda_l=l(l+1)$ with multiplicity $g_l=2l+1$, $l=0,1,2,\ldots$. It is convenient to shift z by 1/4, since this yields the the spectrum of the Dirac operator

$$\sqrt{\lambda_l+\frac{1}{4}}=l+\frac{1}{2}. \tag{18}$$

Then the regularized (shifted) Fredholm determinant

$$\Delta(z)=\prod_{l=0}^\infty\left[\left(1-\frac{z}{(l+\frac{1}{2})^2}\right)e^{z/(l+1/2)^2}\right]^{2l+1} \tag{19}$$

factorizes as

$$\Delta(-y^2)=\Delta^+(iy)\Delta^+(-iy) \tag{20}$$

with the determinant of the Dirac operator $\Delta^+(z)$ given by [13]

$$\Delta^+(z)=\prod_{l=0}^\infty\left[\left(1-\frac{z}{l+\frac{1}{2}}\right)e^{z/(l+1/2)+z^2/2(l+1/2)^2}\right]^{2l+1}. \tag{21}$$

Then this Dirac determinant Δ^+ is related to the Barnes function by

$$\Delta^{+}(z)=\pi^{-1/2}(2\pi)^{z}e^{(1+\gamma+2\log 2)z^2}\frac{\Gamma(\frac{1}{2}-z)G(\frac{1}{2}-z)^2}{G(\frac{1}{2})^2}. \tag{22}$$

We thereby recover the expression relating the moment γ_K to the determinant (13).

This relation between the moments of the distribution and the determinant of the Dirac operator on S^2 is in fact general. For instance, in the simplest case of a single Gaussian random variable, the moments are

$$\int_{-\infty}^{\infty}x^{2K}e^{-x^2}dx=\Gamma(K+\tfrac{1}{2}); \tag{23}$$

$\Gamma(K+1/2)$ is thus the equivalent of γ_K for this trivial problem. If we consider the ''Laplacian,'' i.e., the harmonic oscillator whose eigenvalues are $\lambda_n=n$, then the Fredholm determinant $\Delta(\lambda)$ is

$$\Delta(\lambda)=-\lambda\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n}\right)e^{\lambda/n}$$
$$=\frac{e^{\gamma\lambda}}{\Gamma(-\lambda)}. \tag{24}$$

Hence, we have

$$\langle x^{2K}\rangle\frac{e^{\gamma\lambda}}{\Delta(\lambda)}\bigg|_{\lambda=-(K+1/2)}. \tag{25}$$

The expression (13) is a multivariable version of this Gaussian integral.

An additional point of interest is that the Fredholm determinant of this Laplacian on S^2 may be factorized further into a product of two factors; it turns out that each factor enters into the corresponding expression for the symplectic and orthogonal cases, respectively. This will be seen below when we examine the moments related to the Bessel kernel.

III. BESSEL KERNEL

We have discussed in our previous work the ensembles invariant under the unitary symplectic and unitary orthogonal Lie groups [6]. The kernels for those ensembles are [14–16]

$$K(x,y)=\frac{1}{2\pi}\left(\frac{\sin(x-y)}{x-y}\mp\frac{\sin(x+y)}{x+y}\right), \tag{26}$$

where the minus sign corresponds to the Sp and the plus sign to the O ensemble. It is convenient to introduce the Bessel kernel defined by

$$K_\alpha(x,y)=\frac{J_\alpha(x)J'_\alpha(y)-J'_\alpha(x)J_\alpha(y)}{x-y}. \tag{27}$$

Since $J_{1/2}(x)=\sqrt{2/\pi x}\sin x$, $J_{-1/2}(x)=\sqrt{2/\pi x}\cos x$, the kernels for the Sp and O ensembles are both related to this Bessel kernel

$$K_{\pm}(x,y)=\sqrt{xy}K_{\pm 1/2}(x^2,y^2), \tag{28}$$

namely, $\alpha=1/2$ and $\alpha=-1/2$ represent, respectively, the Sp and the O ensemble. We consider from now on an arbitary α. The $2K$th moment at the origin $(\lambda=0)$ is expressed as

$$I_K=\oint\oint\frac{du\,dv}{(2\pi)^2}\frac{\Delta(u^2)\Delta(v^2)}{\prod_{i=1}^{K}u_i^{2K}v_i^{2K}}\prod_{i=1}^{K}(u_iv_i)^{\alpha}K_\alpha(u_i,v_i). \tag{29}$$

We define now the two functions $\phi(z)$ and $\psi(z)$ by

$$J_\alpha(\sqrt{z})=\left(\frac{\sqrt{z}}{2}\right)^{\alpha}\frac{1}{\Gamma(\alpha+1)}\phi(z) \tag{30}$$

and

$$\sqrt{z}J'_\alpha(\sqrt{z})=\frac{z^{\alpha/2}}{2^{\alpha}\Gamma(\alpha)}\psi(z). \tag{31}$$

Their expansions in powers of x are given by

$$\phi(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n n!\prod_{l=1}^{n}(\alpha+l)} \tag{32}$$

$$\psi(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n(\alpha+2n)}{4^n n!\prod_{l=0}^{n}(\alpha+l)}. \tag{33}$$

Keeping aside trivial factors we are then led to the kernel $\tilde{K}_\alpha(x,y)$ defined as

$$\tilde{K}_\alpha(x,y)=\frac{1}{2(x-y)}[\phi(x)\psi(y)-\psi(x)\phi(y)]. \tag{34}$$

As before, we have

$$I_K=\det(a_{nm}) \tag{35}$$

with

$$a_{nm}=\frac{1}{n!m!}\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m}\tilde{K}_\alpha(u,v)\bigg|_{u=v=0}. \tag{36}$$

This determinant may be computed explicitly, and it is given by

$$I_K=4^{-K^2-\alpha K}\prod_{l=0}^{2K-1}\frac{1}{(\alpha+l)!}. \tag{37}$$

(We have $I_1=\frac{1}{4},1/3\pi,1/\pi$ for $\alpha=0,\frac{1}{2},-\frac{1}{2}$, respectively.)

It is interesting to relate the three determinants that we have introduced above for the sine kernel and for the Sp and O cases. The determinant for the sine kernel (11) is

$$I_{\rm U}=\det\begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 & \dots \\ 0 & \frac{1}{3} & 0 & -\frac{1}{30} & \dots \\ -\frac{1}{6} & 0 & \frac{1}{20} & 0 & \dots \\ 0 & -\frac{1}{30} & 0 & \frac{20}{7!} & \dots \end{pmatrix}. \tag{38}$$

In the symplectic case, $\alpha=\frac{1}{2}$, we have

$$I_{\rm Sp}=\det\begin{pmatrix} \frac{1}{3} & -\frac{1}{30} & \dots \\ -\frac{1}{30} & \frac{20}{7!} & \dots \\ \dots & \dots & \dots \end{pmatrix}. \tag{39}$$

In the orthogonal case, the determinant becomes for $\alpha=-\frac{1}{2}$

$$I_{\rm O}=\det\begin{pmatrix} 1 & -\frac{1}{6} & \dots \\ -\frac{1}{6} & \frac{1}{20} & \dots \\ \dots & \dots & \dots \end{pmatrix}. \tag{40}$$

Thus, we find the factorization of Eq. (38) as the product of Eqs. (39) and (40), up to a trivial numerical factor due to the normalizations,

$$I_{\rm U}=I_{\rm Sp}\times I_{\rm O}. \tag{41}$$

The factors γ_K for the unitary, symplectic, and orthogonal cases are related as $2^{K^2-1}\gamma_K^{(\rm U)}=\gamma_K^{(\rm Sp)}\gamma_K^{(\rm O)}$, and $\gamma_K^{(\rm U)}=(\Pi_{l=1}^{K-1}l!)^2/(\Pi_{l=1}^{2K-1}l!)$, $\gamma_K^{(\rm Sp)}=2^{K(K+1)/2}\Pi_{l=1}^{K}l!/\Pi_{l=1}^{K}(2l)!$, and $\gamma_K^{(\rm O)}2^{K(K+1)/2-1}\Pi_{l=1}^{K-1}l!/\Pi_{l=1}^{K-1}(2l)!$. It is again remarkable that, for arbitrary α, γ_K may still be expressed as the Fredholm determinant of the Laplacian, in which the eigenvalues are shifted by the amount α [13].

IV. AIRY KERNEL

When the eigenvalues lie near an edge λ_c of the support of the asymptotic density of states (an edge of Wigner's semicircle in the Gaussian case), in a neighborhood of size $N^{-2/3}$ of that edge, there is a crossover from the sine kernel to the Airy kernel. In terms of the Airy function $\mathrm{Ai}(x)$, defined by

$$\mathrm{Ai}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dz\, e^{(i/3)z^3+izx}, \tag{42}$$

which satisfies the differential equation

$$\mathrm{Ai}''(x)=x\,\mathrm{Ai}(x), \tag{43}$$

one has

$$K(x,y)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)A_i(y)}{x-y}. \tag{44}$$

In Eq. (44) we have used the scaling variables x and y proportional to $N^{2/3}(\lambda-\lambda_c)$.

There are two ways to obtain the moments under consideration. The first one is to write as before

$$I_K=\langle[\det(\lambda_c-X)]^{2K}\rangle=\oint\frac{du}{2\pi i}\frac{\Delta(u)\Delta(v)}{\prod_{i=1}^{K}u_i^K v_i^K}\prod_{i=3D1}^{K}K(u_i,v_i), \tag{45}$$

but in this case, there are three periodic structures due to three valleys of Airy functions, and the result is more complicated. It does not seem to be expressible as simple products of gamma functions. However, we can use a direct method starting with the expression

$$I_K=\langle[\det(\lambda_c-X)]^{2K}\rangle=\frac{1}{(2\pi)^{2K}}\int_{-\infty}^{\infty}dz\,\Delta^2(z)e^{(i/3)\sum_{i=1}^{2K}z_i^3}. \tag{46}$$

This representation is the edge limit $\lambda_l\to 0$ of

$$F_{2K}=\int_{-\infty}^{\infty}\prod dz_l\oint\frac{du_i}{2\pi i}\frac{\Delta(z)\Delta(u)}{\prod_i\prod_l(u_i-\lambda_c+\lambda_l)}\times e^{N\Sigma_1^{2K}[(i/3)z_l^3+iz_lu_l]}. \tag{47}$$

The sums and products over l run from $l=1$ to $l=2K$. The dependence of F_{2K} on N is of order $N^{2K^2/3-K}$.

We may then use the standard orthogonal polynomial method. To the complex measure

$$d\mu=dz\,e^{iz^3/3}, \tag{48}$$

we associate the orthogonal polynomials p_n defined as

$$p_n(x)=x^n+(\text{lower degree}) \tag{49}$$

and

$$\int d\mu\,p_n(x)p_m(x)=h_n\delta_{n,m}. \tag{50}$$

The integral of Eq. (42) is then simply

$$I=K!h_0h_1\cdots h_{K-1}. \tag{51}$$

Note that this looks similar to the partition function of a matrix model, but here it is the partition function of a $K\times K$ matrix model, instead of $N\times N$ (K is finite, since it is the order of the moment that we are considering, whereas N goes to infinity). Therefore for any K this is a completely explicit expression of the moments at the edge. Those coefficients h_n are expressible in terms of ratios of determinants constructed with the moments of the measure:

$$h_n=\frac{d_n}{d_{n-1}} \tag{52}$$

with

$$d_n=\det\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix} \tag{53}$$

in which the m_n are the moments of the measure. Those determinants are constants along antidiagonal lines (Hankel determinants). Then the product $h_0h_1\cdots h_{K-1}$ is reduced to a single determinant. For example, we have for $K=4$

$$h_0h_1h_2h_3=\det\begin{pmatrix} C_1 & -iC_2 & 0 & iC_1 \\ -iC_2 & 0 & iC_1 & 2C_2 \\ 0 & iC_1 & 2C_2 & 0 \\ iC_1 & 2C_2 & 0 & -4C_1 \end{pmatrix} \tag{54}$$

with $C_1=\mathrm{Ai}(0)=3^{-2/3}/\Gamma(2/3)$, $C_2=\mathrm{Ai}'(0)=-3^{-1/3}/\Gamma(1/3)$, since all the moments up to m_6 are easily expressible in terms of m_0 and m_1 alone.

More generally we have

$$m_n=\int z^n\,d\mu=(-i)^n(n-2)(n-5)(n-8)\cdots\tilde{A}_n \tag{55}$$

where $\tilde{A}_n=C_1$ for $n=0$ (modulo 3), $\tilde{A}_n=C_2$ for $n=1$ (modulo 3), and $\tilde{A}_n=0$ for $n=2$ (modulo 3). The last parenthesis of the product in the RHS of Eq. (55) is the rest of the division of $n-2$ by 3. Then, d_n is the determinant of a Hankel matrix, whose matrix elements in the first row are $[\langle z^0\rangle,\langle z\rangle,\langle z^2\rangle,\ldots]=[C_1,-iC_2,0,iC_1,2C_2,0,-4C_1,10iC_2,0,-28iC_1,-80C_2,0,\ldots]$, and all the others are given by the Hankel rule. In this way we obtain sucessively

$$\begin{aligned} h_0&=C_1=0.355\,028\,053,\\ h_0h_1&=C_2^2=0.066\,987\,483,\\ \prod_0^2 h_l&=2C_2^3+C_1^3=0.010\,074\,161,\\ \prod_0^3 h_l&=-8C_1C_2^3-3C_1^4=0.001\,580\,882,\\ \prod_0^4 h_l&=72C_2^5+28C_1^3C_2^2=0.000\,313\,095\,517,\\ \prod_0^5 h_l&=-2160C_2^6-1952C_1^3C_2^3-432C_1^6\\ &=0.000\,090\,756\,324. \end{aligned} \tag{56}$$

Therefore for the edge problem we have found moments given by γ_K's which are more complicated since $\gamma_K=\Pi_0^{2K-1}h_l$. The result is explicit for any finite K, but we have not succeeded in continuing it to noninteger K. The numerical values indicate a smooth curve in a logarithmic plot.

V. FINITE N RESULTS

We derived in our previous paper [6] the correlation functions of the characteristic polynomials in the form of a determinant,

$$\begin{aligned} F_K(\lambda_1,\ldots,\lambda_K)&=\left\langle\prod_1^K\det(\lambda_l-X)\right\rangle\\ &=\frac{1}{\Delta(\lambda_1,\ldots,\lambda_K)}\det\begin{vmatrix} p_M(\lambda_1) & p_{M+1}(\lambda_1) & \cdots & p_{M+K-1}(\lambda_1)\\ p_M(\lambda_2) & p_{M+1}(\lambda_2) & \cdots & p_{M+K-1}(\lambda_2)\\ \vdots & & & \\ p_M(\lambda_K) & p_{M+1}(\lambda_K) & \cdots & p_{M+K-1}(\lambda_K) \end{vmatrix}, \end{aligned} \tag{57}$$

in which X is an $M\times M$ random matrix.

The polynomials $p_n(x)$ are the (monic) orthogonal polynomials, whose coefficients of highest degree are equal to unity,

$$p_n(x)=x^n+(\text{lower degree}). \tag{58}$$

If we are concerned simply with the moments of the distribution of a single characteristic polynomial, we obtain from Eq. (57)

$$\begin{aligned} \mu_K(\lambda)&=F_K(\lambda,\ldots,\lambda)=\langle[\det(\lambda-X)]^K\rangle\\ &=\frac{(-1)^{K(K-1)/2}}{\prod_{l=0}^{K-1}(l!)}\det\begin{vmatrix} p_M(\lambda) & p_{M+1}(\lambda) & \cdots & p_{M+K-1}(\lambda)\\ p'_M(\lambda) & p'_{M+1}(\lambda) & \cdots & p'_{M+K-1}(\lambda)\\ \vdots & & & \\ p_M^{(K-1)}(\lambda) & p_{M+1}^{(K-1)}(\lambda) & \cdots & p_{M+K-1}^{(K-1)}(\lambda) \end{vmatrix}. \end{aligned} \tag{59}$$

For the Gaussian distribution,

$$P(X)=\frac{1}{Z_M}\exp-\frac{N}{2}\operatorname{Tr}X^2, \tag{60}$$

with

$$M=N-K, \tag{61}$$

the polynomials $p_n(x)$ are the Hermite polynomials $H_n(x)$, defined with our normalization as

$$H_n(x)=\frac{(-1)^n}{N^n}e^{Nx^2/2}\left(\frac{d}{dx}\right)^n e^{-Nx^2/2}=x^n+(\text{lower degree}). \tag{62}$$

The integral representation

$$H_n(x)=\frac{(-1)^n n!}{N^n}\oint\frac{dz}{2i\pi}\frac{e^{-N(z^2/2+xz)}}{z^{(n+1)}} \tag{63}$$

over a contour that circles around the origin in the z plane turns out to be well adapted.

Note that all these expressions are valid for finite N. We may thereby recover readily several results that we have discussed in previous sections. For instance, let us assume that M is an even number, and consider the center value $\lambda=0$ (since the dependence on λ is known to be contained entirely in the overall factor $[\rho(\lambda)]^{K^2}$, as far as the coefficient γ_K is concerned, it is sufficient to put simply $\lambda=0$).

The Hermite polynomials $H_n(x)$ vanish for odd n at $x=0$. Similarly, the odd derivatives of $H_n(x)$ for even n also vanish at $x=0$. Hence, the elements of the determinant (59) are alternately nonzero and then zero. Thus the determinant is decomposed into a product of two determinants; this is the exact phenomenon for finite N of the factorization of the symplectic and orthogonal determinants that we saw earlier for large N. Since the matrix elements of Eq. (59) at $\lambda=0$ are all expressed as derivatives of Hermite polynomials at the origin, it is possible to compute this determinant exactly for finite and arbitrary M and K. For the even M case,

$$F_{2K}(0)=\frac{(-1)^{K(2K-1)}}{\prod_{l=3D0}^{2K-1}(l!)}\det\begin{vmatrix} H_M(0) & H_{M+2}(0) & \cdots \\ H''_M(0) & H''_{M+2}(0) & \cdots \\ \vdots & & \\ H_M^{(2K-2)}(0) & H_{M+2}^{(2K-2)}(0) & \cdots \end{vmatrix}\det\begin{vmatrix} H'_{M+1}(0) & H'_{M+3}(0) & \cdots \\ H'''_{M+1}(0) & H'''_{M+3}(0) & \cdots \\ \vdots & & \\ H_{M+1}^{(2K-1)}(0) & H_{M+3}^{(2K-1)}(0) & \cdots \end{vmatrix}. \tag{64}$$

We denote each determinant as $I^{(1)}/N^{KM/2}$ and $I^{(2)}/N^{KM/2}$, respectively, and

$$F_{2K}(0)=I^{(1)}I^{(2)}\frac{1}{N^{KM}}\frac{1}{\prod_{l=0}^{2K-1}l!}. \tag{65}$$

The above two determinants are easily computed through the explicit expressions for the $H_n(x)$'s,

$$\begin{aligned}H_{2n}(x)=&\frac{1}{n}(-1)^n(2n-1)!!\\ &\times\sum_{m=0}^{\infty}\frac{(-n)(-n+1)\cdots(-n+m-1)}{(\frac{1}{2})(\frac{1}{2}+1)\cdots(\frac{1}{2}+m-1)}\\ &\times\frac{1}{m!}\left(\frac{Nx^2}{2}\right)^m,\end{aligned} \tag{66}$$

$$\begin{aligned}H_{2n+1}(x)=&\frac{1}{N^n}(-1)^n(2n+1)!!x\\ &\times\sum_{m=0}^{\infty}\frac{(-n)(-n+1)\cdots(-n+m-1)}{(\frac{3}{2})(\frac{3}{2}+1)\cdots(\frac{3}{2}+m-1)}\\ &\times\frac{1}{m!}\left(\frac{Nx^2}{2}\right)^m.\end{aligned} \tag{67}$$

The two determinants contain overall products of factors of the form $(2n-1)!!$; once they are extracted one finds

$$\begin{aligned}I^{(1)}&=C(M+2K-3)!!(M+2K-5)!!\cdots(M-1)!!\\ &=C\frac{1}{2^{[K(M+K-3)]/2}}\prod_{l=1}^{K}\left(\frac{\Gamma(M+2l-2)}{\Gamma(M/2+l-1)}\right),\end{aligned} \tag{68}$$

$$\begin{aligned}I^{(2)}&=C(M+2K-1)!!(M+2K-3)!!\cdots(M+1)!!\\ &=C\frac{1}{2^{[K(M+K-1)]/2}}\prod_{l=1}^{K}\left(\frac{\Gamma(M+2l)}{\Gamma(M/2+l)}\right),\end{aligned} \tag{69}$$

with

$$C=2^{K(K-1)/2}\prod_{l=0}^{K-1}l!, \tag{70}$$

which is independent of M. In the large M limit, from the Stirling formula, we have

$$I^{(1)}\simeq CM^{[MK+K(K-1)]/2}e^{-MK/2}2^{K/2}, \tag{71}$$

$$I^{(2)}\simeq CM^{[MK+K(K+1)]/2}e^{-MK/2}2^{K/2}. \tag{72}$$

It is remarkable that, even for finite M (M is the size of the random matrix), $F_{2K}(0)$ for this Gaussian distribution already exhibits the factor $\gamma_K=\Pi_{l=0}^{K-1}l!/(K+l)! =(\Pi_{l=0}^{K-1}l!)^2/\Pi_{l=0}^{2K-1}l!$, which is known to be universal in the large M limit. It is indeed obtained from the product of the factor C and $1/(\Pi_{l=0}^{2K-1}l!)$ in Eq. (65). This means that, at each order of the $1/N$ expansion, we keep this universal factor for $F_{2K}(0)$. In the large N limit ($M=N-K$), $F_{2K}(0)$ becomes

$$F_{2K}(0)\simeq(2N)^{K^2}e^{-NK}\prod_0^{K-1}\frac{l!}{(K+l)!}. \tag{73}$$

In the previous paper, we derived $F_{2K}(\lambda)$, in the large N limit, as

$$F_{2K}(\lambda,\ldots,\lambda)\simeq[2\pi\rho(\lambda)N]^{K^2}e^{-NK}\prod_0^{K-1}\frac{l!}{(K+l)!}. \tag{74}$$

At the band center, $\lambda=0$, the density of state is $\rho(0)=1/\pi$ for the Gaussian distribution. Therefore, Eq. (73) is indeed consistent with Eq. (74).

It may be interesting to note that one of the factors of Eq. (74), namely, $\Pi_{l=0}^{2K-1}(l!)$, appears in $F_{2K}(\lambda)$ in Eq. (59). This factor, a product of gamma functions, remains for any set of orthogonal polynomials, since it stands in front of the determinant of Eq. (59).

The factor e^{-NK} is cancelled by the normalization [6]. For $\lambda\neq0$, we have evaluated $F_{2K}(\lambda)$. We have here considered the finite N case to see the universal factor γ_K.

One can recover again the Airy limit by the use of Eq. (59). We use once more the properties of the Hermite polynomials, such as

$$H_n'(x)=nH(x), \tag{75}$$

and their explicit integral representation,

$$H_n(x)=\frac{1}{\sqrt{2\pi}}N^{1/2}e^{Nx^2/2}\int_{-\infty}^{\infty}ds\, s^n e^{-Ns^2/2-iNxs}. \tag{76}$$

We set $n=\delta+N$, and after exponentiation we have

$$H_n(x)=\frac{1}{\sqrt{2\pi}}N^{1/2}e^{Nx^2/2}\int_{-\infty}^{\infty}ds\, s^{\delta}e^{-Nf(s)}, \tag{77}$$

where $f(s)=\frac{1}{2}s^2+isx-\log s$. The saddle points are degenerate at the edge $x=2$. The vicinity of this point is blown out through a change of variables, with a scaling ansatz

$$x=2+N^{-\alpha}y \tag{78}$$

and

$$s=-i+N^{-\beta}z. \tag{79}$$

If one expands $f(s)$ up to order z^3, one sees that in the proper scaling choice $\alpha=2/3$ and $\beta=1/3$, one recovers the Airy limit which governs the properties of the system in a neighborhood of size $N^{-2/3}$ of the edge of Wigner's semicircle. Then the integral becomes

$$I=(-i)^{\delta}N^{-1/3}\int_{-\infty}^{\infty}dy\, e^{iy^3/3+izy} \tag{80}$$

This is indeed the Airy function $\mathrm{Ai}(z)$ of Eq. (42),

$$H_{N+\delta}(x)=\sqrt{2\pi N}e^{2N}(-i)^{\delta}\mathrm{Ai}((x-2)N^{2/3}). \tag{81}$$

We now consider all the $\lambda_i=2$, and the determinant (59) becomes in the large N limit a determinant of Airy functions. If we replace H_{M+2K-1} at the right top corner of the determinant by the Airy function $\mathrm{Ai}(0)$, the other matrix elements become derivatives of the Airy function, since there is a recursion relation (75). For example, in the $K=1$ case, we have

$$\det\begin{vmatrix}H_M(2) & H_{M+1}(2)\\ H_M'(2) & H_{M+1}'(2)\end{vmatrix}\sim\det\begin{vmatrix}\frac{N^{2/3}}{M+1}\mathrm{Ai}'(0) & \mathrm{Ai}(0)\\ \frac{N^{4/3}}{M+1}\mathrm{Ai}''(0) & N^{2/3}\mathrm{Ai}'(0)\end{vmatrix}. \tag{82}$$

Then we find in the large N limit, with $N=M-K$,

$$F_{2K}(2)=\frac{N^{2K(K+1)/3}}{\prod_{l=0}^{2K-1}l!}\det\begin{vmatrix}\cdots & \mathrm{Ai}'(0) & \mathrm{Ai}(0)\\ \cdots & \mathrm{Ai}''(0) & \mathrm{Ai}'(0)\\ \cdots & \cdots & \cdots\end{vmatrix}. \tag{83}$$

The above determinant was discussed earlier. Note the factor $1/\Pi_{l=0}^{2K-1}l!$ in front.

VI. DERIVATIVE MOMENTS

The same techniques may also be used if one is interested in the moments of the Dth derivatives ($D=1,2,\ldots$) of the characteristic polynomials. Let us consider, for instance,

$$F_{2K}^{(D)}(\lambda_1,\ldots,\lambda_{2K})=\left\langle\prod_{l=3D1}^{2K}\frac{\partial^D}{\partial\lambda_i^D}\det(\lambda_i-X)\right\rangle. \tag{84}$$

From Eq. (59), one sees immediately that it also has the form of a determinant:

$$F_{2K}^{(D)}(\lambda_1,\ldots,\lambda_{2K})=\frac{1}{\Delta(\lambda_1,\ldots,\lambda_{2K})}\det\begin{vmatrix}p_M^{(D)}(\lambda_1) & p_{M+1}^{(D)}(\lambda_1) & \cdots & p_{M+2K-1}^{(D)}(\lambda_1)\\ p_M(\lambda_2)^{(D)} & p_{M+1}^{(D)}(\lambda_2) & \cdots & p_{M+2K-1}^{(D)}(\lambda_2)\\ \vdots & & & \\ p_M^{(D)}(\lambda_{2K}) & p_{M+1}^{(D)}(\lambda_{2K}) & \cdots & p_{M+2K-1}^{(D)}(\lambda_{2K})\end{vmatrix}. \tag{85}$$

When all the λ_i's are equal, we have

$$F_{2K}^{(D)}(\lambda,\ldots,\lambda)=\left\langle\left(\frac{d^D}{d\lambda^D}\det(\lambda-X)\right)^{2K}\right\rangle$$
$$=\frac{(-1)^{K(2K-1)}}{\prod_{l=0}^{2K-1}(l!)}\det\begin{vmatrix} p_M^{(D)}(\lambda) & p_{M+1}^{(D)}(\lambda) & \cdots & p_{M+2K-1}^{(D)}(\lambda)\\ p_M^{(D+1)}(\lambda) & p_{M+1}^{(D+1)}(\lambda) & \cdots & p_{M+2K-1}^{(D+1)}(\lambda)\\ \vdots & & & \\ p_M^{(D+2K-1)}(\lambda) & p_{M+1}^{(D+2K-1)}(\lambda) & \cdots & p_{M+2K-1}^{(D+2K-1)}(\lambda)\end{vmatrix}. \tag{86}$$

If we set $\lambda=0$ it may be again decomposed into a product of two determinants. Let us assume for definiteness that both M and D are even. Then we have

$$I^{(1)}=\det\begin{vmatrix} H_M^{(D)}(0) & H_{M+2}^{(D)}(0) & \cdots\\ H_M^{(D+2)}(0) & H_{M+2}^{(D+2)}(0) & \cdots\\ \vdots & & \\ H_M^{(D+2K-2)}(0) & H_{M+2}^{(D+2K-2)}(0) & \cdots\end{vmatrix}, \tag{87}$$

$$I^{(2)}=\det\begin{vmatrix} H_{M+1}^{(D+1)}(0) & H_{M+3}^{(D+1)}(0) & \cdots\\ H_{M+1}^{(D+3)}(0) & H_{M+3}^{(D+3)}(0) & \cdots\\ \vdots & & \\ H_{M+1}^{(D+2K-1)}(0) & H_{M+3}^{(D+2K-1)}(0) & \cdots\end{vmatrix}. \tag{88}$$

Using the explicit expressions for the Hermite polynomials, we can compute these determinants. We find for arbitrary M, D, and K,

$$F_{2K}^{(D)}(0)=\frac{1}{N^{K(M-D)}}I^{(1)}I^{(2)}\frac{1}{\prod_{l=0}^{2K-1}l!}, \tag{89}$$

$$I^{(1)}=(M+2K-3)!!(M+2K-5)!!\cdots(M-1)!!$$
$$\times\prod_{l=0}^{K-1}\left[\left(\frac{M}{2}+l\right)\left(\frac{M}{2}+l-1\right)\cdots\left(\frac{M}{2}-\frac{D}{2}+l+1\right)\right]$$
$$\times 2^{[DK+K(K-1)]/2}\prod_{l=0}^{K-1}l!, \tag{90}$$

$$I^{(2)}=(M+2K-1)!!(M+2K-3)!!\cdots(M+1)!!$$
$$\times\prod_{l=0}^{K-1}\left[\left(\frac{M}{2}+l+1\right)\left(\frac{M}{2}+l\right)\cdots\left(\frac{M}{2}-\frac{D}{2}+l+2\right)\right]$$
$$\times 2^{[DK+K(K-1)]/2}\prod_{l=0}^{K-1}l!. \tag{91}$$

[One may easily check these results for $D=M$, since the matrix elements below the diagonal vanish, i.e., the determinants are then simply given by the product of the diagonal elements $\Pi_{l=0}^{K}(M+2l)!$ which agrees with Eq. (90). When $D=0$, it reduces to the previous expression (68). $I^{(2)}$ is obtained from $I^{(1)}$ by the shift $M\to M+2$.]

In the large N limit, we have

$$F_{2K}^{(D)}(0)\simeq(2N)^{K^2+2KD}e^{-KN}\frac{1}{2^{2KD}}\prod_{l=0}^{K-1}\frac{l!}{(K+l)!}. \tag{92}$$

Hence, for these derivative moments at finite M, again the universal factor γ_K is present, and it persists, of course, in the large N limit.

These results lead to the conjecture that the average of the moment of derivatives of the Riemann zeta function along the critical line

$$I=\frac{1}{T}\int_0^T dt\left|\frac{d^D}{dt^D}\zeta\left(\frac{1}{2}+it\right)\right|^{2K} \tag{93}$$

also have this universal factor γ_K.

VII. EXTERNAL SOURCE

We now consider the case in which the external source matrix A is coupled to the random matrix X. The measure of the random matrix X is

$$d\mu(X)=\frac{1}{Z}e^{-N\,\mathrm{Tr}\,X^2/2+N\,\mathrm{Tr}\,XA}d^{N^2}X. \tag{94}$$

The eigenvalues of the matrix A are denoted by a_i, $i=1,\ldots,N$. In such cases, the standard orthogonal polynomial method cannot be used. However, the n-point correlation functions $\rho(\lambda_1,\ldots,\lambda_n)$ have been found to be described again by the determinant of a kernel; from there the level spacing probability $p(s)$ has also been investigated [9]. If we specialize to a source that has two opposite eigenvalues, namely, $a_i=+a$ for $i=1,\ldots,N/2$ and $a_i=-a$ for $i=N/2+1,\ldots,N$, we find a support for the eigenvalues made of two disconnected segments for $a>1$. If one tunes the external source so that $a=1$, i.e., $a_i=\pm1$, the gap between the two segments closes and the spectrum consists of a single segment for $a<1$. We want to investigate here the critical point $a=1$ which gives rise to yet another class of universality. The moments $F_{2K}(\lambda,\ldots,\lambda)$ at $\lambda=0$ at this closing gap point may turn out to have interesting applications.

Since X and A are Hermitian matrices, we write

$$\mathrm{Tr}\,XA=\mathrm{Tr}\,U^{-1}X_0UA_0, \tag{95}$$

where $X_0=\mathrm{diag}(x_1,\ldots,x_M)$, $A_0=\mathrm{diag}(a_1,\ldots,a_M)$, and U belongs to the unitary group. The integration over this unitary group U is well known from the work of Harish-Chandra and Itzykson and Zuber [18,19], and this is the starting point of the formulas found in [8]. For instance, the n-point correlation functions are given by the determinant of the $n\times n$ matrices made with the kernel $K(\lambda_i,\lambda_j)$ with

$$K(\lambda,\mu)=\int_{-\infty}^{\infty}\frac{dt}{2\pi}\oint\frac{du}{2\pi i}\prod_{l=1}^{N}\frac{a_l-it}{u-a_l}\frac{1}{u-it}\times e^{-Nt^2/2+Nit\lambda-Nu^2/2+Nu\mu+N\lambda^2/4-N\mu^2/4} \quad (96)$$

where the contour encloses all the a_l's.

However, we may proceed without that here and compute the correlation functions of the characteristic polynomials directly. Indeed,

$$F_K(\lambda_1,\cdots,\lambda_K)=\left\langle\prod_{\alpha=1}^{K}\det(\lambda_\alpha-X)\right\rangle=\frac{1}{Z}\int dX\prod_{\alpha=1}^{K}\det(\lambda_\alpha-X)\times e^{-N\,\mathrm{Tr}\,X^2/2+N\,\mathrm{Tr}\,XA}. \quad (97)$$

In the above equation, the random matrix X is assumed to be an $M\times M$ matrix, with $M=N-K$, as before. When $K=1$, this gives a polynomial, which was investigated before [17].

Explicit integration over the unitary group [18,19] leads to

$$F_K(\lambda_1,\ldots,\lambda_K)=\int\prod_{i=1}^{M}dx_i\frac{\Delta(x_1,\ldots,x_M;\lambda_1,\ldots,\lambda_K)}{\Delta(a)\Delta(\lambda)}\times e^{-(N/2)\sum_{i=1}^{M}x_i^2+N\sum_{i=1}^{M}x_ia_i} \quad (98)$$

where $\Delta(x_1,\ldots,x_M;\lambda_1,\ldots,\lambda_K)$ is the Van der Monde determinant $(M+K)\times(M+K)$ made with the x's and the λ's. This determinant may be replaced by a determinant of (monic) polynomials, and we choose the Hermite polynomials defined in Eq. (62). It is then straightforward to verify that

$$\int_{-\infty}^{\infty}H_n(x)e^{-Nx^2/2+Nax}dx=a^ne^{Na^2/2}\sqrt{\frac{2\pi}{N}}. \quad (99)$$

Therefore we can explicitly integrate over the M variables x_i in Eq. (98) and obtain

$$F_K(\lambda_1,\ldots,\lambda_K)=\frac{1}{\Delta(a)\Delta(\lambda)}\det\begin{vmatrix}1&\cdots&1&H_0(\lambda_1)&\ldots&H_0(\lambda_K)\\ \vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\ a_1^{M+K-1}&\ldots&a_M^{M+K-1}&H_{M+K-1}(\lambda_1)&\ldots&H_{M+K-1}(\lambda_K)\end{vmatrix}. \quad (100)$$

Let us first check that in the limit of a vanishing source in which all the $a_i\to 0$ we do recover the previous formula (57). The column that depends upon a_i is expanded in Taylor series around a_1, and, subtracting the successive columns, we obtain, after factoring the Van der Monde determinant $\Delta(a)$ which cancels the denominator, a vanishing upper triangle (up to the Mth column), 1s on the diagonal and powers of the a_i's below the diagonal. We can now let the a_i's go to zero and we are left with the $K\times K$ of Eq. (57). (In [6] we gave a different derivation of this same formula.)

If we return to an arbitrary nonvanishing external source, we may proceed by returning to Eq. (100) and define $G_K(b_1,\ldots,b_K)$,

$$G_K(b_1,\ldots,b_K)=\int F_K(\lambda_1,\ldots,\lambda_K)\Delta(\lambda)\times e^{-(N/2)\Sigma\lambda_l^2+N\sum\lambda_lb_l}\prod d\lambda_i=\frac{\Delta(a;b)}{\Delta(a)}e^{(N/2)\Sigma b_l^2}. \quad (101)$$

We may now recover F_K by taking the Fourier transform of $G_K(ib_1,\ldots,ib_K)$,

$$\int G_K(ib_1,\ldots,ib_K)e^{-iN\Sigma\lambda_ib_i}\prod_{i=1}^{K}\frac{db_i}{2\pi}=\left(\frac{1}{N}\right)^K\Delta(\lambda)F_K(\lambda_1,\ldots,\lambda_K)e^{-(N/2)\Sigma\lambda_l^2}. \quad (102)$$

Therefore, we obtain the explicit formula,

$$F_K(\lambda_1,\ldots,\lambda_K)=\frac{N^K}{\Delta(\lambda)}e^{(N/2)\Sigma\lambda_l^2}\frac{1}{K!}\int\prod_{i=1}^{K}\frac{db_i}{2\pi}\times\prod_{j=1}^{M}(ib_l-a_j)\prod_{l<l'}^{K}(ib_l-ib_{l'})\times e^{-(N/2)\Sigma b_l^2}\det(e^{-iN\lambda_lb_{l'}}). \quad (103)$$

Note that we could replace $\det(e^{-iN\lambda_lb_{l'}})$ in the integrand of Eq. (103) by the diagonal term $e^{-iN\Sigma_1^K\lambda_lb_l}$ and cancel the $K!$ in the denominator. Again we can examine the limit of this formula when the external source goes to zero, and putting all $\lambda_i=\lambda$ we obtain

$$F_{2K}(\lambda)=\frac{N^{K(2K+1)}}{\prod_{l=0}^{2K-1} l!}\frac{1}{(2K)!}e^{KN\lambda^2}\int\prod_{l=1}^{2K} b_l^M\Delta^2(b)\times e^{-(N/2)\Sigma b_l^2-iN\lambda\sum b_l}\prod_{l=1}^{2K}\frac{db_l}{2\pi} \quad (104)$$

(we have considered F_{2K} instead of F_K in order to compare with our previous results). In the large N limit, we exponentiate $b_l^M (M=N-K)$, and look for the saddle points which are the roots of the equation $b^2+i\lambda b-1=0$; let us call the two roots b^+ and b^-. The difference $|b^+-b^-|=2\pi\rho(\lambda)$. The leading saddle point for the b_l's ($l=1,\ldots,2K$) is obtained by choosing half of them equal to b^+ and b^- for the another half. The Gaussian integral with a Van der Monde determinant,

$$\frac{1}{K!}\int\prod_{i=1}^{K} db_i e^{-Nf''b^2/2}\prod_{i<j}^{K}(b_i-b_j)^2 = \left(\frac{2\pi}{Nf''}\right)^{K/2}\frac{\prod_{l=0}^{K-1} l!}{(Nf'')^{K(K-1)/2}}, \quad (105)$$

where f'' is the second derivative of f at the saddle point, allows us to complete the calculation. Integrating then around the saddle points b^+ and b^-, and keeping in mind the combinatorial factor $(2K)!/K!K!$, which is the number of choices of Kb^+ and Kb^- among the $2Kb_l$'s, we recover precisely our previous result,

$$e^{-NKV(\lambda)}F_{2K}(\lambda)=[2\pi N\rho(\lambda)]^{K^2}e^{-NK}\gamma_K, \quad (106)$$

where

$$\gamma_K=(\Pi_{l=0}^{K-1}l!)^2/(\Pi_{l=0}^{2K-1}l!)=(\Pi_{l=0}^{K-1}l!)/\Pi_{l=0}^{K-1}(K+l)!,$$

and $V(\lambda)=\lambda^2/2$.

From the expression (103), it is also easy to obtain the moments at the critical point corresponding to the closure of the gap:

$$F_K(0)=\frac{N^K}{K!}e^{NM/2}\int\prod_{l=1}^{K}\frac{db_l}{2\pi}(1+b_l^2)^{M/2}\Delta^2(b)e^{-(N/2)\sum_{l=1}^{K} b_l^2}. \quad (107)$$

Note that this expression is exact for finite N. In the large N limit, we exponentiate the logarithmic term and expand the exponent about b_l up to the order b_l^4 term. The critical point is precisely the point at which the coefficient of the quadratic term b_l^2 vanishes. We then have

$$F_K(0)=\frac{N^K}{K!}e^{NM/2}\int\prod\frac{db_l}{2\pi}e^{-(N/2)\sum_{l=1}^{K} b_l^4}\Delta^2(b). \quad (108)$$

As in all the cases that appeared in the previous sections, this integral is expressed by a Hankel determinant, in which the matrix elements are $\Gamma((2n-1)/4)$. The determinant is

$$I=\det\begin{vmatrix}\Gamma(\frac{1}{4}) & 0 & \Gamma(\frac{3}{4}) & 0 & \cdots\\ 0 & \Gamma(\frac{3}{4}) & 0 & \Gamma(\frac{5}{4}) & \cdots\\ \Gamma(\frac{3}{4}) & 0 & \Gamma(\frac{5}{4}) & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{vmatrix}. \quad (109)$$

Note that we have considered the case $a_l=\pm 1$, but the formulas are explicit for any spectrum of the source and they could be easily used to study, for instance, the multicritical situations that were discussed in [9].

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